Numerical Integration

The "Equations of Uniformly Accelerated Motion" are valuable tools for specifying the motion of many objects, but many real objects—including elevators, cars, rockets and more—move in ways that are more complex. Whether acceleration is uniform or not, it is still possible to shift back and forth between relationships which show position as a function of time, $x = f(t)$, and those which show velocity as a function of time, $v = f(t)$. As Isaac Newton recognized, instantaneous velocity is the rate at which position is changing, or

$$v = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

Numerical integration is an important application of this relationship, used by scientists, computer programmers, engineers and others to deal with mathematical functions which are difficult or impossible to solve by other methods. The process almost always involves an approximation, but modern

Skylon Tower, Niagara Falls
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computer techniques can produce results which are very close to the exact theoretical results. The method is a recursive calculation, based on the fact that when Δt is very small compared to the rate at which velocity is changing,

$$v = \frac{dx}{dt} \cong \frac{\Delta x}{\Delta t}$$

Multiplying both sides by Δt, $v\,\Delta t = \frac{dx}{dt}\,\Delta t \cong \Delta x$

Rearranging, $\Delta x \cong \frac{dx}{dt}\,\Delta t = v\,\Delta t$

Therefore, $x = x_{old} + \Delta x = x_{old} + \frac{dx}{dt}\Delta t$

Or, $x = x_{old} + v * \Delta t$

If we know the initial position, x0, we can find a close approximation to each successive position, x, by adding v * Δt, even it v changes in complex, non-uniform ways. The process must be repeated many times, updating the calculation often enough that v is almost constant during each calculation interval.

Task: In this activity you will use information about velocity and the method of numerical integration to specify a variety of simple and complex motions.
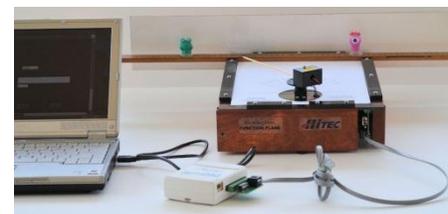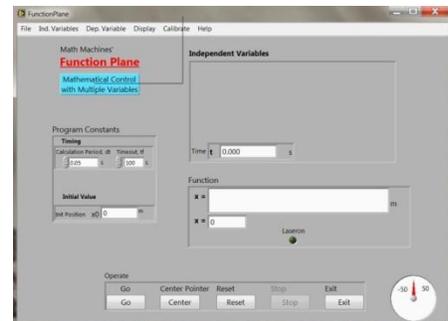
Additional Materials: Screen with 2 targets.

Math Machines Program:

Function Plane

Activity Files: NumIntegration00x

This activity uses a Class II diode laser, similar to those used in many barcode scanners. Never look directly into the laser beam and never allow it to shine into anyone's eyes.

1. <u>Uniform Velocity</u>. To explore and test the method of numerical integration, first consider an example of uniform velocity, where the laser dot begins at x = -25 cm and moves to x = 20 cm in 10 seconds. Load Activity File "NumIntegration001" to begin the task.

   a. What is the initial position?   x0 = _____.cm

   b. What is the uniform velocity?   v = _____.cm/s

   c. Verify that the correct *explicit* function, x = f(t) is entered in the function box and that the "Timeout" value is set to 10 s, then run the motion. Does the laser dot actually move from -25 cm to 20 cm in 10 seconds?

      _____

   d. Now recreate the same motion using a *recursive* function, where x = f(x_{old}).

   i.   Verify that the initial position, x0, is set to -25 cm and the calculation period, dt, is set to 0.05 s.

      ii. Delete the explicit function, x = f(t), and replace it with a recursive function in the form
           "x = x_{old} + v * Δt". This computer program cannot easily accept either Greek letters or subscripts, so the required syntax is shown below. Modify the function by substituting the value you found in step *1 b* for v. Enter your function into the computer function box and in the space below.

           Required syntax: x = "xold + v * dt"

           Your function: x = _____

   e. Since velocity is uniform, your recursive function should give precisely the same result as the original explicit function. Is that what happened?

      _____

   f. Finding the final position with the method of numerical integration is equivalent to finding the "Area under the Curve" for a graph of velocity vs. time like the one at right. Show below how you can obtain the same result for the <u>total change in position of x</u> (from t = 0 s to t = 10 s) from this graph.

Velocity as a Function of Time
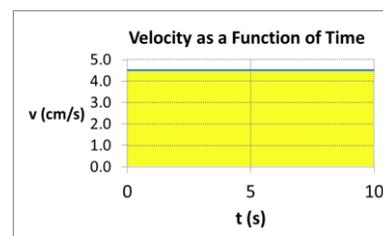
2. <u>Uniform Acceleration</u>. Load Activity File NumIntegration002, which shows an *explicit* function, x = f(t), to produce an accelerated motion beginning at -25 cm, moving to 20 cm in 10 seconds where it stops for an instant, and then returning to the starting point after an additional 10 seconds.

   a. What is the initial position?   x0 = _____.cm

   b. What is the initial velocity?   v0 = _____.cm/s

   c. What is the acceleration?   a = _____.cm/s^2

d. Verify that the correct *explicit* function, x = f(t), is entered in the function box and that the "Timeout" value is set to 20 s, then run the motion. Does the laser dot actually move from -25 cm to 20 cm and back in 20 seconds?

_____

e. Now recreate the same motion using a *recursive* function, where $x_{new} = f(x_{old})$.

   i.   Delete the *explicit* function, x = f(t), and replace it with a *recursive* function in the form
        "x = xold + v * dt". Unlike the previous case, v is not constant. Write the function below which shows v as a function of time, v -= f(t).

        v = _____

   ii. Insert your function for v into a recursive function, x = xold + v * dt, to create your recursive function. Write the function below and enter it into the computer's functions box.

        x = _____

   iii. Run the motion. The result should be very close to the results of the explicit function, but it may not be precisely the same. What differences, if any, do you see between the results of the explicit and recursive functions?

        _____

   iv. Increase the calculation period, dt, from 0.05 s to 0.10 s and rerun the motion again with your same recursive function. Does the error change? Explain your answer.
        _____

        _____

   v.   Try decreasing the calculation period as a way of decreasing error. What happens if you decrease the calculation period too much?
   _____

        _____

f. There are a number of variations on numerical integration which STEM professionals use to improve precision without requiring more computer time or more expensive hardware. For example, to predict the position when t = 1.00 s, it is better to use the velocity at the center of the preceding calculation interval rather than using the velocity when t = 1.00 s. This improved method can be implemented here by using "(t-dt/2)" instead of "t" when calculating velocity. Set your calculation period back to 0.05 s and modify your recursive function to use "(t-dt/2)" instead of "t". Run the program again and describe below how this influences the precision of the result.

g. Show below how you can obtain the same result for the
    change in position of x by finding the "area under the
   curve" for each of the time intervals below:

    Velocity as a Function of Time

    i.   0 s ≤ t ≤ 10 s

    ii.   10 s ≤ t ≤ 20 s

    iii. 0 s ≤ t ≤ 20 s
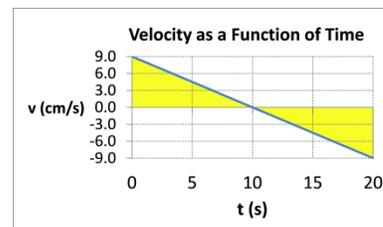
3. Elevator. Load Activity File NumIntegration003 and consider a
    case where the velocity changes as a function of time as shown
    in the graph at right. This pattern might occur, for example, in an
    elevator which moves upwards from one floor to another in 25
    seconds. Mathematically, $v = f(t) = 0.5\,t - 0.02\,t^2$.
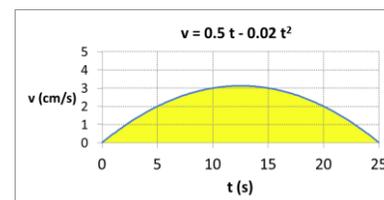
    a. Do the equations of uniformly accelerated motion work with
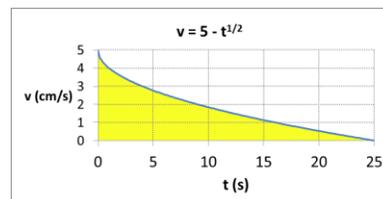   this velocity? Explain your answer.

    _____

    b. Design and test a recursive function, $x = x_{old} + v\,dt$, which will create this motion. As
        before, let
        x0 = -25 cm. Write the function below and enter it into the computer's functions box.

        X = _____

    c. Run the motion for 25 s. Describe below how the dot moves.

    _____

    _____

    d. Can you check your result for the final position of the dot by finding the "area under the
        curve" for the graph? Explain your answer.

    _____

    _____

4. <u>Uneven Braking</u>. Load Activity File NumIntegration004 and consider a case where the velocity changes as a function of time as shown in the graph at right. This pattern might occur, for example, if a driver initially hits the brakes hard then gradually eases off while her car stops. Mathematically, v = f(t) = 5 − t$^{1/2}$.[1]

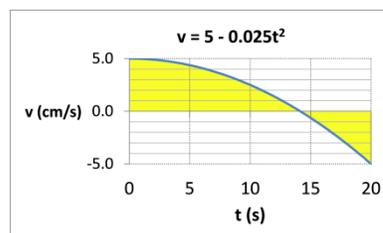   a. Do the equations of uniformly accelerated motion work with this velocity? Explain your answer.

   _____

   b. Design and test a recursive function, x = x$_{old}$ + v dt, which will create this motion. Again let x0 = -25 cm. Write the function below and enter it into the computer's functions box.

   X = _____

   c. Run the motion for 25 s. Describe below how the dot moves.

   _____

   _____

   d. Can you check your result for the final position of the dot by finding the "area under the curve" for the graph? Explain your answer.

   _____

   _____

5. <u>Ship at Sea</u>. Load Activity File NumIntegration005 and consider a case where the velocity changes as a function of time as shown in the graph at right. This pattern might occur, for example, if an oil tanker were moving forward when the Captain reverses engines, producing a backwards net force that increases over time. Mathematically, v = f(t) = 5 − 0.025 t^2

   a. Do the equations of uniformly accelerated motion work with this velocity? Explain your answer.

   _____

   b. Design and test a recursive function, x = x$_{old}$ + v dt, which will create this motion. Write the function below and enter it into the computer's functions box.

   X = _____

   c. Run the motion for 20 s. Describe below how the dot moves.

   _____

   _____

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[1] The square root of t can be entered into the Function Box as either "t^.5" or "t^(1/2)" or "sqrt(t)".

d. The velocity begins at +5 cm/s and ends at -5 cm/s. The formula $v_{average} = (v_i + v_f)/2$ seems to indicate that the average velocity is zero and the final position should be the same as the initial position. Explain below why the average velocity is <u>not</u> zero.

6. <u>Additional Cases</u>. Test as many of the velocity functions below as you can. For each, record the recursive function you design, run it for 25 seconds and describe the resulting motion in words. Except where indicated otherwise, keep x0 = -20.

   a. $v = t^3 / 1000$

   b. $v = 8 – 0.8 * t$

   c. $v = 20 * \sin(2\pi * t / 5)$ [Enter as "20*sin(pi(2)*t/5)"]

   d. $v = -x/5$

   e. $v = (x-20)/5$

   f. Repeat 6d, but with x0 = + 5 cm.

   g. Load the Activity File NumIntegration006g, run it and adjust the slider values, r1. Describe the resulting motions and explain how the recursive function makes it happen.